SEPARATION PROPERTIES

RECALL: If $(X, \mathcal{T})$ is a topological space and $A \subseteq X$, then

$$\overline{A} = \{x \in X \,:\, x \in T \in \mathcal{T} \implies T \cap A \neq 0\}$$

In some sense, the points in $\overline{A}$ are points that are in A or topologically 'close' to A.

In this section, we introduce notions of separating points and sets.

DEFINITION: Let $(X, \mathcal{T})$ be a topological space and let x and y be distinct elements of X. We say $(X, \mathcal{T})$ is:

- T_0 (or a Kolmogorov Space) if there exists $T \in \mathcal{T}$ so that $x \in T$ and $y \notin T$ or $y \in T$ and $x \notin T$.
- T_1 (or a Fréchet Space) if there exist $T, U \in \mathcal{T}$ so that $x \in T$ and $y \notin T$ and $y \in U$ and $x \notin U$.
- T_2 (or Hausdorff) if there exist $T, U \in \mathcal{T}$, $T \cap U = \emptyset$ so that $x \in T$ and $y \in U$.

EXAMPLE: Refer back to the 29 topologies on the set $\{a, b, c\}$. Which are T_0? T_1? T_2?

EXAMPLE: When is a set with the indiscrete topology ever T_0? T_1? T_2?

EXAMPLE: When is a set with the discrete topology ever T_0? T_1? T_2?

EXAMPLE: Show $(\mathcal{R}, \mathcal{E})$ and $(\mathcal{R}, \mathcal{S})$ are both T_2.

EXAMPLE: Prove if $(X, \mathcal{T})$ is T_1, then $\{x\}$ is closed for all $x \in X$. Is the converse true?

EXAMPLE: Prove $T_2 \implies T_1 \implies T_0$.

EXAMPLE: Prove T_0, T_1, and T_2 are **hereditary** properties. That is, these properties are inherited by subspaces.

EXAMPLE: Is the continuous image of a T_0 space necessarily T_0? What if T_0 is replaced with T_1 or T_2?

What if 'continuous' is replaced with 'open' or 'closed'?

EXAMPLE: T_0, T_1, and T_2 are **productive** properties. That is, is the product of two T_0 spaces T_0, etc.

DEFINITION: Given a set X, the **diagonal** , D of $X \times X$ is the set $D = \{(x, x) \,:\, x \in X\}$.

THEOREM: $(X, \mathcal{T})$ is Hausdorff if and only if D is closed in $(X \times X, \mathcal{T} \times \mathcal{T})$.

PROJECT IDEA: There are many more separation properties involving separating points from closed sets and disjoint closed sets. Define these and give examples.